Exam Sets & Numbers (part 2: integers and modular arithmetic),
November 9th, 2022, 8:30am–10:30am,
Aletta Jacobshal 04 H18-M17.

Write your name on every sheet of paper that you intend to hand in. Please provide **complete** *arguments for each of your answers. The exam consists of* 3 *questions. You can score up to* 9 *points for each question, and you obtain* 3 *points for free. In this way you will score in total between* 3 *and* 30 *points.*

(1) A question regarding gcd's:
  (a) [3 points] Find $a, b, x, y \in \mathbb{Z}$ with the property $a^2x + b^2y = 42$.
  (b) [3 points] Show that if $a, b, x, y \in \mathbb{Z}$ satisfy $a^2x + b^2y = 42$, then $\gcd(a, b) = 1$.
  (c) [3 points] Show that if $a, b \in \mathbb{Z}$ satisfy $\gcd(a, b) = 1$, then $x, y \in \mathbb{Z}$ exist such that $a^2x + b^2y = 42$.

(2) A function appearing in this exercise, is $f\colon \mathbb{Z}_{\geq 1} \to \mathbb{Z}/35\mathbb{Z}$ given by $f(n) = \overline{2}^n$.
  (a) [1 point] Explain why $\overline{5}$ is not in the image of the function f.
  (b) [3 points] Find $m \geq 1$ such that for every $k \in \mathbb{Z}_{\geq 0}$ and for every $n \in \mathbb{Z}_{\geq 1}$ we have $f(n + mk) = f(n)$.
  (c) [3 points] Prove that $35|(2^{9112022} + 31)$.
  (d) [2 points] Explain why $7 \cdot 2^n \equiv 7^{n+1} \bmod 35$ holds for every $n \in \mathbb{Z}_{\geq 0}$.

(3) For any integer $n \geq 1$ we define $a_n = \sum_{j=0}^{n-1} 100^j$. So $a_1 = 1$ and $a_2 = 101$ and $a_3 = 10101$ and $a_4 = 1010101$, et cetera. These numbers satisfy (if necessary you may use this without proving it) $a_{n+1} = 100a_n + 1$ and $a_{n+1} = a_n + 100^n$ for every $n \geq 1$.
  (a) [3 points] Find *all* $m \in \mathbb{Z}_{\geq 1}$ with the property that the subset $\{\overline{a_n} \,:\, n \in \mathbb{Z}_{\geq 1}\} \subseteq \mathbb{Z}/m\mathbb{Z}$ consists of only one element.
  (b) [2 points] Find n such that $33|a_n$.
  (c) [2 points] Prove that if $p \notin \{2, 3, 5, 11\}$ is a prime number, then
$$p|a_{(p-1)/2}.$$
  (d) [2 points] Show that for $n \neq 2$, the integer a_n is *not* a prime number.